FAMILIES OF ARCS IN E^3

BY

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1. **Introduction.** In [6], R. H. Bing announced the result that Euclidean 3-space does not contain an uncountable collection of pairwise disjoint wild surfaces. J. Stallings, in [19], gave an example of an uncountable collection of pairwise disjoint wild disks in 3-space. In [17] Stallings' construction has been modified so as to obtain an uncountable collection of pairwise disjoint inequivalently imbedded disks. In §4 of this paper we consider arcs in 3-space which are locally tame modulo a compact 0-dimensional set. Examples of such arcs have been studied in [1], [3], [9], [10], [11], and [12]. It is shown (assuming the Continuum Hypothesis) that an uncountable collection of pairwise disjoint arcs of this type exists which contains a representative of each imbedding class (under space homeomorphism) of such arcs.

In §5 we answer, in the affirmative, the following question raised in [8] by Bing: Given a 2-sphere S in E^3, is there a family F of mutually exclusive tame arcs such that for each point of S at which S can be pierced by a tame arc, there is a member of F piercing there?

2. **Definitions and notation.** We use E^n to denote Euclidean n-space, coordinatized in the usual way by a set of mutually perpendicular axes $X_1, X_2, \ldots,$ and X_n, and with points denoted by their coordinates $(x_1, x_2, \ldots, x_n)$. If $p=(x_1, x_2, \ldots, x_n)$ and $q=(y_1, y_2, \ldots, y_n)$, the distance between p and q is $\rho(p,q)=(\sum_{i=1}^{n}(x_i-y_i)^2)^{1/2}$. If $p \in E^n$ and ε is a positive number, $N(p,\varepsilon)$ is the set $\{p' \in E^n \mid \rho(p,p')<\varepsilon\}$. If X is a bounded subset of E^n, the diameter of X is the least upper bound of the set of numbers $\{\rho(p,q) \mid p, q \in X\}$. An ε-set is a subset of E^n whose diameter is no more than ε. We use $D_{\max}(X)$ to denote the least upper bound of the set of numbers which are diameters of components of X. If X is a subset of the x_1x_2-coordinate plane in E^3, $X \times [a,b]$ denotes the set $\{(x_1,x_2,x_3) \mid (x_1,x_2,0) \in X \text{ and } a \leqq x_3 \leqq b\}$. The homeomorphism $h\colon E^n \to E^n$ is an ε-homeomorphism if $\rho(p,h(p))<\varepsilon$ for each $p \in E^n$.

An arc A in E^3 is *tame* if there is a homeomorphism of E^3 onto itself taking A onto a polygonal arc. Otherwise, A is *wild.* The arc A is *locally tame* (*locally polygonal*) at the point $p \in A$ if p lies in the interior of a tame subarc (polygonal subarc) of A. Otherwise A is *locally wild* (*locally nonpolygonal*) at p. If X is a closed subset of A and A is locally tame (locally polygonal) at each point of $A-X$, then

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A is said to be *locally tame* (*locally polygonal*) *modulo* X. For generalizations of the above concepts, see §1 of [4].

A compact 0-dimensional subset M of E^3 is *tame* if there is a homeomorphism of E^3 onto itself taking M into a polygonal arc. Otherwise M is *wild*. See [5] for characterizations of tame compact 0-dimensional subsets of E^3.

The closure of a set X will be denoted by Cl (X). The notation Int (X) may mean point set interior, combinatorial interior, or bounded complementary domain. Similarly, Bd (X) may mean point set boundary or combinatorial boundary. When seen in context, the usage will be clear in each case.

The setting for the results included here, with one exception, is E^3. The exception is Theorem 5 which generalizes Theorem 4 to E^n, $n>3$.

3. **Preliminary results.** In this section some results will be obtained dealing with imbeddings of compact 0-dimensional sets in E^3, one of which, Theorem 1, will find application in the following sections.

Suppose C is a polyhedral cube with $n \geqq 0$ handles in E^3. Then C is the homeomorphic image of a closed neighborhood of a bouquet of n circles in E^3. If $n=0$, interpret "bouquet of 0 circles" as "point". The image of the bouquet under such a homeomorphism shall be referred to as a *center* of C. We shall refer to any bouquet of n simple closed curves in C which is equivalently imbedded in C (under a homeomorphism of C onto itself fixed on Bd (C)) to a center of C as a *central curve* of C. Notice that if K is a closed subset of Int (C), B is a central curve of C, and $\varepsilon>0$, then there is a homeomorphism of C onto itself, fixed on Bd (C), carrying K into $N(B, \varepsilon)$.

LEMMA 1. *If C is a polyhedral cube with n handles, and A is an arc, then there is a central curve S of C so that $A \cap S=\varnothing$.*

Proof. There is a homeomorphism h of C into E^3 taking B, a center of C, into the x_1x_2-coordinate plane. Let $\varepsilon>0$ be a number so small that $(h(B)\times[0, \varepsilon]) \subset \text{Int}\,(h(C))$. There is a positive number $\delta \leqq \varepsilon$ so that the plane $x_3=\delta$ meets the set $h(A \cap C)$ in at most a set of dimension 0, for an arc cannot contain uncountably many pairwise disjoint open intervals. Using elementary facts from plane topology, one is able to construct a central curve for $h(C)$, lying in the plane $x_3=\delta$, which misses $h(A \cap C)$. The inverse image of this curve, under h, is a central curve of C which misses A.

THEOREM 1. *Suppose M is a compact 0-dimensional subset of E^3, $\{A_i\}_{i=1}^{\infty}$ is a countable collection of arcs, and $\varepsilon>0$. Then there is an ε-homeomorphism $h\colon E^3 \to E^3$ such that $h(M) \cap (\bigcup_{i=1}^{\infty} A_i)=\varnothing$.*

Proof. The set M is definable by cubes with handles (Lemma 4 of [2] or Theorem 2 of [15]). In other words, there is a sequence $M_1, M_2, M_3, \ldots$ of compact 3-manifolds with boundary such that

(1) for each positive integer i, M_i is the union of a finite number of pairwise disjoint polyhedral cubes with handles,

(2) for each positive integer i, $M_{i+1} \subset \operatorname{Int}(M_i)$, and

(3) $M = \bigcap_{i=1}^{\infty} M_i$.

The homeomorphism h will be the limit of homeomorphisms $h_1, h_2, h_3, \ldots$. We begin the construction of the sequence in this paragraph with the construction of h_1. Since M is totally disconnected, there is a positive integer $i(1)$ such that $D_{\max}(M_{i(1)}) < \max(1, \varepsilon)$. Denote the components of $M_{i(1)}$ by $C_1^1, \ldots, C_{n(1)}^1$. If $j = 1, 2, \ldots$, or $n(1)$, C_j^1 is a polyhedral cube with handles. By Lemma 1, C_j^1 has a central curve S_j^1 such that $S_j^1 \cap A_1 = \varnothing$. Let f_j^1 be a homeomorphism of C_j^1 onto itself, fixed on $\operatorname{Bd}(C_j^1)$, which moves $M_{i(1)+1} \cap C_j^1$ so close to S_j^1 that $f_j^1(M_{i(1)+1} \cap C_j^1) \cap A_1 = \varnothing$. Let $h_1 \colon E^3 \to E^3$ be defined by $h_1(p) = p$ if $p \in E^3 - M_{i(1)}$, $h_1(p) = f_j^1(p)$ if $p \in C_j^1$.

Now suppose that homeomorphisms $h_1, \ldots, h_m$ of E^3 onto itself have been defined so that

(4) there are positive integers $i(1) < i(2) < \cdots < i(m)$ such that $h_j | E^3 - M_{i(j)} = h_{j-1} | E^3 - M_{i(j)}$, where $j = 1, 2, \ldots$, or m and h_0 is the identity on E^3,

(5) $D_{\max}(h_j(M_{i(j)})) < 1/j$, if $j = 1, 2, \ldots$, or m, and

(6) $h_j(M_{i(j)+1}) \cap (\bigcup_{i=1}^{j} A_i) = \varnothing$, if $j = 1, 2, \ldots$, or m.

Let $i(m+1)$ be an integer greater than $i(m)$ such that $D_{\max}(h_m(M_{i(m+1)})) < 1/m+1$. Denote the components of $h_m(M_{i(m+1)})$ by $C_1^{m+1}, \ldots, C_{n(m+1)}^{m+1}$. If $j = 1, 2, \ldots$, or $n(m+1)$, C_j^{m+1} is a cube with handles and $C_j^{m+1} \cap (\bigcup_{i=1}^{m} A_i) = \varnothing$. By Lemma 1, C_j^{m+1} has a central curve S_j^{m+1} such that $S_j^{m+1} \cap A_{m+1} = \varnothing$. Let f_j^{m+1} be a homeomorphism of C_j^{m+1} onto itself, fixed on $\operatorname{Bd}(C_j^{m+1})$, which moves $h_m(M_{i(m+1)+1}) \cap C_j^{m+1}$ so close to S_j^{m+1} that $f_j^{m+1}(h_m(M_{i(m+1)+1}) \cap C_j^{m+1}) \cap A_{m+1} = \varnothing$. Let $h_{m+1} \colon E^3 \to E^3$ be defined by $h_{m+1}(p) = h_m(p)$ if $p \in E^3 - M_{i(m+1)}$, $h_{m+1}(p) = f_j^{m+1}(h_m(p))$ if $p \in h_m^{-1}(C_j^{m+1})$. The homeomorphisms $h_1, \ldots, h_{m+1}$ satisfy conditions (4)–(6) above, so the construction of the sequence may be continued in the fashion described here.

It is not difficult to show that $h = \lim \{h_i\}$ is a homeomorphism. The set M is carried by h onto the set $\bigcap_{i=1}^{\infty} h(M_i)$. For each positive integer j, $h(M_{i(j)+1}) = h_j(M_{i(j)+1})$ fails to intersect $\bigcup_{i=1}^{j} A_i$, so $h(M)$ fails to intersect $\bigcup_{i=1}^{\infty} A_i$. Since h acts as the identity on $E^3 - M_{i(1)}$ and $D_{\max}(M_{i(1)}) < \varepsilon$, h is an ε-homeomorphism.

COROLLARY 1. *Suppose M is a compact 0-dimensional subset of E^3, $\varepsilon > 0$, and A is a subset of E^3 which is contained in the union of a countable collection of arcs. Then there is an ε-homeomorphism $h \colon E^3 \to E^3$ such that $h(M) \cap A = \varnothing$.*

An example of a set A satisfying the hypothesis of Corollary 1 is the union of all straight lines which are parallel to a coordinate axis and intersect a coordinate plane in a point all of whose coordinates are rational. Applying Corollary 1, we obtain the following result.

COROLLARY 2. *If M is a compact 0-dimensional subset of E^3, and $\varepsilon>0$, then there is an ε-homeomorphism h: $E^3 \to E^3$ such that if $(x_1, x_2, x_3) \in h(M)$, then at most one of the numbers x_1, x_2, x_3 is rational.*

One might wonder if the conclusion of Corollary 2 could be strengthened to conclude that all of the numbers x_1, x_2, x_3 are irrational. The following shows that, in general, it can not be.

THEOREM 2. *If M is a compact 0-dimensional subset of E^3, then M is tame if and only if there is a homeomorphism h: $E^3 \to E^3$ such that if $(x_1, x_2, x_3) \in h(M)$, then the numbers x_1, x_2, x_3 are irrational.*

Proof. If M is tame, there is a homeomorphism h_1: $E^3 \to E^3$ such that $h_1(M) \subset L$, where L is the straight line through the point $(\pi, \pi, 0)$ parallel to the x_3-axis. By the analog of Theorem 1 on the real line, there is a homeomorphism h_2: $L \to L$ such that if $(\pi, \pi, x_3) \in h_2h_1(M)$, then x_3 is irrational. Let h_3 be a homeomorphism of E^3 onto itself such that h_3 restricted to L is h_2. Then $h=h_3h_1$ carries M onto a set of points of the form (π, π, x_3) with x_3 irrational.

Now suppose M is compact and 0-dimensional and that h is a homeomorphism of E^3 onto itself such that $(x_1, x_2, x_3) \in h(M)$ implies x_1, x_2, x_3 are irrational. Let $(x_1, x_2, x_3) \in h(M)$ and ε be an arbitrary positive number. If i is 1, 2 or 3, let r_1^i and r_2^i be rational numbers such that $x_i-\varepsilon/4<r_1^i<x_i<r_2^i<x_i+\varepsilon/4$. The set $C=\{(x_1', x_2', x_3') \mid r_1^i \leqq x_i' \leqq r_2^i,\ i=1, 2, 3\}$ is a polyhedral cube of diameter less than ε containing (x_1, x_2, x_3) in its interior and having no point of $h(M)$ on its boundary. By Theorem 3.1 of [5], $h(M)$, and hence M, is tame.

COROLLARY 3. *If M is a compact 0-dimensional subset of E^3 and the set of points of M which have a rational coordinate is closed and tame, then M is tame.*

Proof. Let $N=\{(x_1, x_2, x_3) \in M \mid x_1$ or x_2 or x_3 is rational$\}$. By Theorem 2, M is locally tame modulo N. By Theorem 6.1 of [5], M is tame.

LEMMA 2. *Suppose $\{M_i\}_{i=1}^{\infty}$ is a countable collection of compact 0-dimensional sets in E^2, a and b are points of $E^2-\bigcup_{i=1}^{\infty} M_i$, and J is a simple closed curve containing a and b. Then there is an arc ab such that* $\operatorname{Int}(ab) \subset \operatorname{Int}(J)$ *and* $ab \cap (\bigcup_{i=1}^{\infty} M_i)=\varnothing$.

REMARK. The proof of Lemma 2 to be given here involves the concept of simple chains. For definitions of this and other terms, one is referred to Chapter 3, pp. 105–119, of [14].

Proof. By the classical Schoenflies Theorem, we may suppose that J is the unit circle in E^2. Let A_1 be a polygonal arc from a to b so that $\operatorname{Int}(A_1) \subset \operatorname{Int}(J)$ and $A_1 \cap M_1=\varnothing$. As a convention, let us suppose that all arcs with end points a and b are linearly ordered from a to b. Then A_1 is the union of a finite collection $A_1^1, \ldots, A_{n(1)}^1$, of straight line intervals such that

(1) if $j=1, 2, \ldots,$ or $n(1)$, then the diameter of A_j^1 is less than 1,

(2) the intersection of any pair of intervals $A_1^1, \ldots, A_{n(1)}^1$ is either empty or an end point of each, and

(3) if $1 \leqq j < k \leqq n(1)$, then $p \in \operatorname{Int}(A_j^1)$ implies that p precedes each point of A_k^1.

If $j=1, 2, \ldots,$ or $n(1)$, there is an open disk d_j^1 containing A_j^1 such that

(4) $C_1: d_1^1, \ldots, d_{n(1)}^1$ is a simple 1-chain from a to b,

(5) if $j=2, 3, \ldots,$ or $n(1)-1$, $d_j^1 \subset \operatorname{Int}(J)$,

(6) $(\bigcup_{i=1}^{n(1)} d_i^1) \cap M_1 = \varnothing$, and

(7) if $j=1, 2, \ldots,$ or $n(1)-1$, $d_j^1 \cap d_{j+1}^1$ is an open disk.

There is a polygonal arc A_2 from a to b such that

(8) $\operatorname{Int}(A_2) \subset \operatorname{Int}(J)$,

(9) $A_2 \subset \bigcup_{i=1}^{n(1)} d_i^1$,

(10) if $p \in A_2$ precedes $q \in A_2$ and p and q are in the same link of C_1, then every point of A_2 between p and q also lies in the link, and

(11) $A_2 \cap M_2 = \varnothing$.

The arc A_2 is the union of a finite collection $A_1^2, \ldots, A_{n(2)}^2$ of straight line intervals such that

(12) if $j=1, 2, \ldots,$ or $n(2)$, then the diameter of A_j^2 is less than $1/2$,

(13) the intersection of any pair of intervals $A_1^2, \ldots, A_{n(2)}^2$ is either empty or an end point of each,

(14) if $1 \leqq j < k \leqq n(2)$, then $p \in \operatorname{Int}(A_j^2)$ implies that p precedes each point of A_k^2, and

(15) if $j=1, 2, \ldots,$ or $n(2)$, there is an integer k such that $A_j^2 \subset d_k^1$.

If $j=1, 2, \ldots,$ or $n(2)$, there is an open disk d_j^2 containing A_j^2 such that

(16) $C_2: d_1^2, \ldots, d_{n(2)}^2$ is a simple $1/2$-chain from a to b,

(17) if $j=2, 3, \ldots,$ or $n(2)-1$, $d_j^2 \subset \operatorname{Int}(J)$,

(18) $(\bigcup_{i=1}^{n(2)} d_i^2) \cap M_2 = \varnothing$,

(19) if $j=1, 2, \ldots,$ or $n(2)-1$, $d_j^2 \cap d_{j+1}^2$ is an open disk, and

(20) C_2 is a refinement of C_1 running straight through C_1.

Continuing in this manner, we construct simple chains $C_1, C_2, C_3, \ldots$ such that

(21) $C_k: d_1^k, \ldots, d_{n(k)}^k$ is a simple $1/k$-chain from a to b,

(22) if $j=2, 3, \ldots,$ or $n(k)-1$, $d_j^k \subset \operatorname{Int}(J)$,

(23) $(\bigcup_{i=1}^{n(k)} d_i^k) \cap M_k = \varnothing$, and

(24) C_{k+1} is a refinement of C_k running straight through C_k. By the proof of Theorem 3-15, pp. 116–117 of [14], $ab = \bigcap_{k=1}^{\infty} (\bigcup_{i=1}^{n(k)} d_i^k)$ is an arc from a to b. By condition (22) satisfied by the simple chains $C_1, C_2, C_3, \ldots$, $\operatorname{Int}(ab) \subset \operatorname{Int}(J)$, while by condition (23), $ab \cap (\bigcup_{i=1}^{\infty} M_i) = \varnothing$.

The proof of Lemma 2 shows that, in $E^n (n \geqq 2)$, the complement of a countable union of compact sets of dimension less than $n-1$ is arcwise connected. An example given by R. H. Bing in the case $n=3$ (Theorem 6.3 of [5]) shows that there may be no polygonal arcs in this set.

4. **Arcs locally tame modulo a compact 0-dimensional set.** If A is an arc in E^3, the *wild set* of A, denoted by $W(A)$, is the set $\{p \in A \mid A \text{ is locally wild at } p\}$. The *tame set* of A is the set $T(A)=A-W(A)$. Note that $W(A)$ is closed, $T(A)$ is open (relative to A) and A is locally tame modulo $W(A)$. In this section we shall consider only those arcs A for which $W(A)$ is a compact 0-dimensional subset of E^3.

Subsets H and K of E^3 are said to be *equivalently imbedded* in E^3 if there is a homeomorphism $h: E^3 \to E^3$ such that $h(H)=K$. This notion induces an equivalence relation on the nonempty subsets of E^3, where the sets H and K are equivalent if and only if H and K are equivalently imbedded in E^3. We use $C(H)$ to denote the equivalence class of H under this equivalence relation. It is easy to see that if A is an arc and $B \in C(A)$, then $W(B) \in C(W(A))$. The converse is, in general, false. By [12], there are uncountably many equivalence classes of arcs which are locally tame modulo an end point. By [11], there are uncountably many equivalence classes of arcs which are locally tame modulo an interior point. Since every Cantor set M lies on an arc which is locally tame modulo M, it follows from [18] that there are uncountably many equivalence classes of arcs which are locally tame modulo a Cantor set.

The result to be proved in this section is that there is a collection $\{A_\alpha\}$ of pairwise disjoint arcs in E^3 so that if A is an arc in E^3 with a compact 0-dimensional wild set, then some member of $\{A_\alpha\}$ is a member of $C(A)$.

THEOREM 3. *There is a collection $\{A_\alpha\}$ of pairwise disjoint arcs in E^3 such that if A is an arc in E^3 with $W(A)$ a compact 0-dimensional set, then there is an $A' \in \{A_\alpha\}$ such that $C(A')=C(A)$.*

Proof. The theorem is to be proved by giving a constructive process for building the collection $\{A_\alpha\}$. Assuming the Continuum Hypothesis, the disjoint equivalence classes of arcs locally tame modulo a compact 0-dimensional set can be well-ordered so that each class is preceded in the ordering by only countably many classes. The construction of $\{A_\alpha\}$ is begun by simply choosing a representative of the first class in the well-ordered sequence. To show that the construction may be continued through the sequence, suppose $C(B)$ is an equivalence class of arcs locally tame modulo a compact 0-dimensional set and that the arcs $\{A_i\}_{i=1}^\infty$ have been constructed as pairwise disjoint representatives of the classes preceding $C(B)$; a member of $C(B)$ shall be constructed in $E^3-(\bigcup_{i=1}^\infty A_i)$. The element of $C(B)$ in $E^3-(\bigcup_{i=1}^\infty A_i)$ will be constructed by adjusting B as described below.

By Theorem 1, there is no loss of generality in supposing that $W(B)\subset E^3-(\bigcup_{i=1}^\infty A_i)$. By Theorem 1 of [16] we may also suppose that B is locally polygonal modulo $W(B)$. Then $T(B)$ is the union of a countable number of straight line intervals. It is not hard to show that there is a homeomorphism of E^3 onto itself which is fixed on $W(B)$, moves intervals of $T(B)$ onto intervals, and end points of intervals of $T(B)$ into $E^3-(\bigcup_{i=1}^\infty A_i)$. We denote the image of B under this homeomorphism by B'.

Denote the intervals of $T(B')$ by $I_1, I_2, I_3, \ldots$. There is a null-sequence of polyhedral 2-spheres $S_1, S_2, S_3, \ldots$ such that

(1) $S_n \cap B'$ consists of the end points of I_n,

(2) $\text{Int}(I_n) \subset \text{Int}(S_n)$, and

(3) $S_n \cap S_m = \varnothing$ if $I_n \cap I_m = \varnothing$, $S_n \cap S_m$ is the common end point of I_n and I_m if $I_n \cap I_m \neq \varnothing$.

For each n, there is a plane P_n containing I_n and intersecting each element of $\{A_i\}_{i=1}^{\infty}$ in at most a compact 0-dimensional set. Denote the end points of I_n by a_n and b_n. Let H_n be an arc from a_n to b_n lying in $P_n \cap (\text{Int}(S_n) \cup \{a_n, b_n\})$ and so that $(\text{Int}(H_n)) \cap I_n = \varnothing$. By Lemma 2, there is an arc K_n in P_n so that (1) $K_n \cap (\bigcup_{i=1}^{\infty} A_i) = \varnothing$, and (2) $\text{Int}(K_n) \subset \text{Int}(I_n \cup H_n)$. It is clear that the arc $(B - \bigcup_{n=1}^{\infty} I_n) \cup (\bigcup_{n=1}^{\infty} K_n)$ lies in $C(B)$ and misses $\bigcup_{i=1}^{\infty} A_i$.

5. **Piercing 2-spheres with tame arcs.** The 2-sphere S in E^3 is said to be *pierced* by the arc ab at the point $p \in S$ if

(1) $S \cap (ab) = p$,

(2) $p \in \text{Int}(ab)$, and

(3) a and b lie in different complementary domains of S.

Since each point of S is arcwise accessible from both $\text{Int}(S)$ and $\text{Ext}(S)$, S can be pierced by an arc at each of its points. However, an example described in [10] shows that there may be points of S at which it is impossible to pierce S with a tame arc.

In [8], Bing showed that each 2-sphere in E^3 can be pierced by a tame arc. David Gillman showed, in [13], that the set of points at which a 2-sphere cannot be pierced by a tame arc is a 0-dimensional F_σ set.

Theorem 4 below provides an affirmative answer to a question raised by Bing in [8].

THEOREM 4. *Given a 2-sphere S in E^3, there is a family F of mutually exclusive tame arcs such that for each point of S at which S can be pierced by a tame arc, there is a member of F piercing there.*

Proof. The collection F is to be constructed in much the same way as the collection $\{A_\alpha\}$ was constructed in the proof of Theorem 3. The crucial step is in proving that the following statement is true: If $\{A_i\}_{i=1}^{\infty}$ is a countable collection of pairwise disjoint tame arcs each of which pierces S, and p is a point of $S - (\bigcup_{i=1}^{\infty} A_i)$ at which S can be pierced by a tame arc, then there is a tame arc $A \subset E^3 - (\bigcup_{i=1}^{\infty} A_i)$ which pierces S at p.

Under the above conditions, let ab be a tame arc which pierces S at p and whose end points, a and b, lie in $E^3 - (\bigcup_{i=1}^{\infty} A_i)$. The point a is supposed, without loss of generality, to lie in $\text{Ext}(S)$. Let $h\colon E^3 \to E^3$ be a homeomorphism of E^3 onto itself carrying ab onto a straight line interval. There is a plane P containing $h(ab)$ so that P intersects each element of $\{A_i\}_{i=1}^{\infty}$ in at most a 0-dimensional set.

Let A' be an arc in $P \cap (\text{Ext}(h(S)))$ from $h(a)$ to $h(p)$ so that $A' \cap (\bigcup_{i=1}^{\infty} h(A_i)) = \varnothing$ and let A'' be an arc in $P \cap (\text{Int}(h(S)))$ from $h(b)$ to $h(p)$ so that $A'' \cap (\bigcup_{i=1}^{\infty} h(A_i)) = \varnothing$. Since $A' \cup A''$ lies in P, $A' \cup A''$ is a tame arc which pierces $h(S)$ at $h(p)$ and which lies in $E^3 - (\bigcup_{i=1}^{\infty} h(A_i))$. Let $A = h^{-1}(A' \cup A'')$.

If S is an $(n-1)$-sphere in E^n, and $n > 3$, then S can be pierced by a tame arc at each of its points. From the above proof we obtain the following

THEOREM 5. *If S is an $(n-1)$-sphere in E^n, and $n > 3$, then there is a family F of mutually exclusive tame arcs such that S is pierced at each of its points by a member of F.*

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